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ments from negatives of portions of the lunar surface, presented by the Lick Observatory; Photographs of the Great Sun-spot of 1897 January, presented by G. J. Newbegin.

Also twenty-five works from the library of the late Mr. Sidney Waters, and an equatorially-mounted camera with telephotographic lens, prepared for the observation of the total eclipse of 1896, presented by Mrs. Waters.

On the Mean Motions of the Lunar Perigee and Node.

By Professor Ernest W. Brown, M.A.

The following is a brief abstract of the paper:

The calculations necessary for obtaining a new development of the lunar theory having progressed to a stage at which it is possible to obtain very approximate values of the mean motions of the perigee and node of the Moon, it is of some interest to see how the results compare with those of Delaunay and Hansen. The motions are now obtained so as to include the second powers of the eccentricities and inclination. The remaining parts are estimated from Delaunay's results. Using Dr. Hill's values for the parts due to the figure of the Earth, the results are:

Mean annual motion of Perigee	$+146\ 435''\!\cdot\!3$,
,, ,, ,, Node	$-69\ 679\cdot5$;

with a possible error not exceeding $2''$ in the case of the perigee, and $1''$ in the case of the node. Hansen's results, when corrected for the part depending on the figure of the Earth, give

$$+146\ 434''\!\cdot\!9,$$
$$-69\ 677\cdot3.$$

The observations give

$$+146\ 435''\!\cdot\!6,$$
$$-69\ 679\cdot5.$$

It appears that the difference of $2''\!\cdot\!2$ which Hansen found in the case of the node is at least partly due to some error. Nothing can be said as yet concerning the question as to whether the Newtonian law is exact, an hypothesis treated by Professor Newcomb in his "Astronomical Constants." He finds that a difference of $+1''\!\cdot\!40$ would result in the case of the perigee. The corrected value of Hansen gives only $+0''\!\cdot\!7$, while that which I have obtained is $+0''\!\cdot\!3$, but the error of the last result may possibly be as great as $2''$. The question awaits the determination of the higher terms due to the Sun's action—a determination which I hope to make in the course of a year or two.

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1. Up to the present time, the only theoretical values of the mean motions of the lunar perigee and node which even pretend to be accurate to one seventy-thousandth part of the whole, are those derived by Hansen. Delaunay's algebraical developments, though carried in some cases to a high power of the ratio of the mean motions, are quite insufficient, and doubts have been thrown on the accuracy of his higher terms.

Apart from the question of the formation of tables for the Moon's motion, there is the interesting one which Professor Newcomb has brought forward in his "Astronomical Constants" as to whether the Newtonian law of gravitation requires any modification. The supposition made is that the force with which one body attracts another does not vary as the exact inverse second power of the distance, but that the index differs from -2 by some very small fraction. By assuming the index to be $-2{\cdot}0000001574$, the differences between theory and observation in the cases of the perihelia of *Mars* and *Mercury* are fairly well accounted for (*op. cit.* p. 119), the effects on those of the other planets not being at present within the range of observation. But the same number will cause an addition to the annual mean motion of the lunar perigee of $+1''{\cdot}40$; the effect on the motion of the nodes is insensible. The comparison of the numbers given by Hansen shows a difference of $+1''{\cdot}56$ for the perigee, but Professor Newcomb adds that the argument loses all sustaining force on account of a difference of nearly $3''$ in the case of the node. Thus, either Hansen's values are not accurate, or some other cause affects the motion of the node.

2. My calculations of the solar inequalities in the Moon's motion having reached a stage at which it is possible to obtain the theoretical values of these mean motions accurately as far as the second order inclusive, in relation to the eccentricities and inclinations, it is interesting to see, first, how Delaunay's algebraical results appear as to accuracy and convergence; secondly, how Hansen's numerical results appear as to accuracy; thirdly, whether any fresh light can be thrown on the hypothesis mentioned above. In the new results given below, the question of convergence, from a practical point of view, does not enter at all; for the results are obtained by using the *numerical* value of the ratio of the mean motions of the Sun and Moon (a ratio known from observation with great accuracy), the values of the other constants only being inserted at the end of the work. The series for the quantities under consideration here when arranged according to powers of the squares of the eccentricities, inclination, and ratio of the parallaxes, diminish so rapidly that it is quite unnecessary to proceed further than the fourth powers of these constants, while those of this order are very small.

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3. In order that the degree of accuracy obtained may be clearly defined, I give below a detailed comparison of the new results with those obtained by Delaunay, and also a comparison of the whole with those derived by Hansen from theory and observation. Delaunay's numerical results being given in terms of the *daily* motions, I use the latter for the discussion of them, returning to *annual* motions when the final general values are to be considered. The word "order" used below refers only to powers of the eccentricities, inclination, and the square root of the ratio of the parallaxes (this ratio being really a small quantity of the second order), and not to powers of the ratio of the mean motions.

4. As the terms of the fourth order are as yet undetermined (using the word "order" in the sense just explained), it is necessary to deduce their values from the results given by Delaunay. That these may be as accurate as possible, before proceeding to the main discussion, I give a test of their accuracy. Estimation is made of the remainders in each case. It is right to state here that these estimations were made in such a way that any conscious or unconscious influence which a knowledge of the result required for agreement with observation would have caused, was eliminated by one or two artifices. The possible errors have intentionally been made large, so that no false idea as to the accuracy of the final results might be entertained.

5. Although no positive evidence for or against the hypothesis treated by Professor Newcomb is obtained, one result seems to be demonstrated, namely, that the value obtained by Hansen for the annual motion of the node is at least one, if not two, seconds in error. Further, if his value for the part of the motion of the perigee which depends on the solar action is correct, one half of the small difference between theory and observation can be accounted for by using Dr. Hill's instead of Hansen's value for the part depending on the figure of the Earth. This, however, tells nothing definite, for if doubts are thrown on the accuracy of Hansen's calculations for the motion of the node, we are not entitled to assume the accuracy of the result for the perigee. The most probable theoretical values for both these motions are given at the end of the paper.

A Test of Delaunay's Expressions.

6. In a paper which will shortly appear in the Proceedings of the London Mathematical Society, the following results are shown to hold when the square of the ratio of the solar to the lunar parallax is neglected.

$$c_1 + c_2 \frac{\partial \pi_1}{\partial n} + c_3 \frac{\partial \theta}{\partial n} = \frac{3}{2}(\mathrm{E}+\mathrm{M}) \frac{\partial}{\partial n}\left(\frac{1}{r}\right)_0,$$

$$c_2 \frac{\partial \pi_1}{\partial \,.\, e^2} + c_3 \frac{\partial \theta_1}{\partial \,.\, e^2} = \frac{3}{2}(\mathrm{E}+\mathrm{M}) \frac{\partial}{\partial \,.\, e^2}\left(\frac{1}{r}\right)_0,$$

$$c_2 \frac{\partial \pi_1}{\partial \,.\, \gamma^2} + c_3 \frac{\partial \theta_1}{\partial \,.\, \gamma^2} = \frac{3}{2}(\mathrm{E}+\mathrm{M}) \frac{\partial}{\partial \,.\, \gamma^2}\left(\frac{1}{r}\right)_0.$$

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Here c_1, c_1+c_2, $c_1+c_2+c_3$ are the values of Delaunay's final variables L, G, H, after all the periodic terms have been eliminated ;* π_1, θ_1 are the mean motions of the perigee and node ; $E+M = n^2a^3$ is the sum of the masses of the Earth and Moon, and $\left(\frac{1}{r}\right)_0$ denotes the constant term in the value of $1/r$ when the latter is exhibited as a sum of periodic terms depending on the time. All the quantities are supposed to be expressed in terms of Delaunay's final constants n, a, e, γ, e', a', n'.

With the help of the expression for the constant part of the parallax given by Adams,† these equations may be used to verify Delaunay's results. The values of L, G, H, and thence those of c_1, c_2, c_3 in terms of the final constants, have been deduced by Professor Newcomb.‡ Since Delaunay does not make use of any similar verification equations, I have thought it worth while to test his results as far as it was possible with Adams' expression. The highest power of m tested is given in each case.

Test of the values of c_1, c_2, c_3, given by Professor Newcomb on p. 184 of the memoir just referred to. The value of c_1 is tested completely except for the part depending on a^2/a'^2. The value of c_2 contains e^2 as a factor of all its terms. It is tested as far as e^2m^3, e^4m^1 ; there are no terms involving $e^2\gamma^2m^0$, $e^2\gamma^2m^1$, $e^2e'^2m^0$, $e^2e'^2m^1$. The value of c_3 contains γ^2 as a factor of all its terms. It is tested as far as γ^2m^3, $\gamma^2e^2m^1$; there are no terms involving $\gamma^2e'^2m^0$, $\gamma^2e'^2m^1$, γ^4m^0, γ^4m^1.

We now come to the motions of the perigee and node. It is only possible to test these to a low power of m in any case. Every term tested was found correct. As the numerical values of the terms which contain e^2, γ^2, e'^2 as factors are given below, no detailed statement is necessary. But as Delaunay's values for the terms of higher orders are used, I give the results obtained.

The equation most convenient for these terms is

$$\frac{\partial c_2}{\partial\cdot\gamma^2}\cdot\frac{\partial\pi_1}{\partial\cdot e^2}+\frac{\partial c_3}{\partial\cdot\gamma^2}\cdot\frac{\partial\theta_1}{\partial\cdot e^2}-\frac{\partial c_2}{\partial\cdot e^2}\cdot\frac{\partial\pi_1}{\partial\cdot\gamma^2}-\frac{\partial c_3}{\partial\cdot e^2}\cdot\frac{\partial\theta_1}{\partial\cdot\gamma^2}=0,$$

which is deducible directly from the second and third of the equations given above.

In the motion of the perigee the terms involving $e^2\gamma^2m^2$, $e^2\gamma^2m^3$, γ^4m^2, γ^4m^3, $\gamma^2e^2m^2$, $\gamma^2e^2m^3$, $\gamma^2e'^2m^2$, $\gamma^2e'^2m^3$, were tested and found correct. In the motion of the node, the terms involving e^4m^2, e^4m^3, $e^2\gamma^2m^2$, $e^2\gamma^2m^3$, $e^2e'^2m^2$, $e^2e'^2m^3$ were tested and found correct. It will be noticed that these terms include all the larger terms given in the tables below for terms of the fourth order, except those depending on a^2/a'^2.

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* *Théorie de la Lune*, vol. ii. pp. 235, 236.

† *M.N.R.A.S.* vol. xxxviii. *Coll. Works*, p. 203.

‡ Papers published for the use of the Amer. Eph. vol. v. pp. 184, 201, 202.

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The Motion of the Perigee.

7. *Term of Zero Order.*—This has been completely obtained and discussed by Dr. G. W. Hill [*] and M. Andoyer. [†] The value which results from the numerical determination made by the former is,

Daily Motion, +406″·6402.

8. *Terms of the Second Order.*—The following table gives the daily motions obtained by Delaunay, [‡] arranged according to the powers of m, the ratio of the mean motions. As usual, e, γ, e' are the constants defined in Delaunay's manner.

Daily Motions.

Factors.	e^2	γ^2	e'^2
	″	″	″
m^2	−0·3000	−3·2085	+0·0840
m^3	− ·6311	− ·9450	+ ·1440
m^4	− ·2763	− ·3705	+ ·0998
m^5	− ·1213	− ·1466	+ ·0539
m^6	− ·0387	...	...
m^7	+ ·0006	...	...
Sum	−1·3668	−4·6706	+0·3817
Est. Rem.	+ ·004	− ·07	+ ·05
Delaunay	−1·36	−4·74	+0·43
True value	−1·4218	−4·7627	+0·4278

The part contributed by the term of any order in relation to e^2, γ^2, e'^2, m^i is seen in this table. Thus the term of order $m^4\gamma^2$, obtained by Delaunay, furnishes −0″·3705. The line "Est. Rem." is the estimated sum of the remaining terms of Delaunay's expressions. The next line gives the resulting value which is thus obtained, and in the last line are the correct values resulting from my calculations. When the possible errors of estimation are taken into account, the agreement is fairly satisfactory, except in the case of the terms depending upon e^2.

The results of the calculations for these latter are given in my paper 'On the Elliptic Inequalities in the Lunar Theory.' [§] Feeling that the discordance was too great to rest on a single determination, I recalculated the whole of the inequalities depending on e^2 by my later method. [‖] An almost exact agreement with the earlier result was found, showing that certainly Delaunay's coefficient for e^2m^7, and perhaps that for e^2m^6, are

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[*] *Acta. Math.* vol. viii. pp. 1–36. *Annals of Math.* vol. ix. pp. 31–41.

[†] *Ann. de la Fac. des Sc. de Toulouse*, vol. vi.

[‡] *C. R.* vol. lxxiv. pp. 19 *et sq.*, where these and the following results will be found.

[§] *Amer. Jour. Math.* vol. xv. pp. 244–263, 321–338.

[‖] *Ibid.* vol. xvii. pp. 318–358.

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seriously wrong. It will be noticed that the small estimated remainder has the positive sign, whereas the true value shows that it should be a comparatively large quantity with the negative sign.

9. *Terms of the Fourth Order.*—The values of these must be obtained from Delaunay's expressions. The manner in which his method is carried out, taken in conjunction with the tests considered in § 6, give us a reasonable assurance of their accuracy; doubt only arises from the estimated remainders.

Daily Motions.

Factors.	e^4	γ^4	e'^4	a^2/a'^2
	″	″	″	″
m^2	−0·0002	−0·0121	+0·0000	+0·0024
m^3	...	+0·0056	...	+0·0019
Est. Rem.	−0·0004	+0·0040	+0·0001	+0·0020
Delaunay	−0·0006	−0·0025	+0·0001	+0·0063

Daily Motions.

Factors.	γ^2e^2	$\gamma^2e'^2$	$e^2e'^2$
	″	″	″
m^2	+0·0139	−0·0013	−0·0001
m^3	+0·0003	−0·0010	−0·0006
Est. Rem.	−0·0020	−0·0010	−0·0010
Delaunay	+0·0122	−0·0033	−0·0017

The sum of these gives

Daily Motion, +0″·0105,

with a possible error* due to wrong estimation of ± 0″·005. It will be noticed that the error arises chiefly from the terms which contain γ^4, γ^2e^2, a^2/a'^2 as factors.

Annual Motion of the Perigee.

10. Gathering together the true values given in §§ 7, 8, and the result obtained in § 9, we multiply them by 365·25 to reduce them to annual motions in a Julian year. The portions due to the figure of the Earth and to planetary action must also be included. The following are the results, with the authorities for them.

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* No technical sense is to be attached to this expression.

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Annual Motions.

Terms of the form $nf(m)$	+148 525″·3	(Hill)	
„ „ „ $ne^2f(m)$	− 519·3	(Brown)	
„ „ „ $n\gamma^2f(m)$	− 1 739·6	„	
„ „ „ $ne'^2f(m)$	+ 156·2	„	
„ „ higher orders	+ 3·8 ± 1″·8	(Delaunay)	
From the solar action	+146 426·4 ± 1·8		
Figure of the Earth	+ 6·8	(Hill)*	
Planetary action	+ 2″·1	(Hansen)†	
From theory	**+146 435·3 ± 1·8**	. . . .	(i)
From observation	**+146 435·6**	(Hansen)† . .	(ii)
From Hansen's theory	**+146 434·0**	„ . . .	(iii)

Hansen, however, has obtained for the inequalities depending on the figure of the Earth the value 5″·9. Accepting the value obtained in the very extended and accurate investigation made by Dr. Hill, we must add + 0″·9 to Hansen's result, making it

+146 434″·9. . . . . (iv)

The evidence of agreement with observation afforded by the comparison of the values (i), (ii), is unfortunately rendered useless by the large possible error of ± 1″·8, which I have felt compelled to assign to the estimations of the higher terms due to the solar action. Hansen's determination of the part due to planetary action is probably correct within two or three tenths of a second.

If Hansen's result for the solar part be correct and we accept the value (iv), which at this stage is the best theoretical determination, it would seem that the hypothesis considered by Professor Newcomb is hardly verified in the case of the Moon. The required difference is 1″·4, while that furnished by the values (ii) and (iv) above is only 0″·7. The question must remain unanswered until an exact determination of the solar inequalities has been made. Such a determination is now in progress, and I hope may be finished before the lapse of any very considerable time, seeing that the outstanding parts, which are the chief cause of the possible error, are those easiest to calculate.

Motion of the Node.

11. *Term of Zero Order.*—This has been calculated to a high degree of accuracy by Professor J. C. Adams ‡ and Mr. P. H.

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* Papers published for the Amer. Eph. vol. iii. p. 344.

† Hansen's results given here and in § 14, are found in Arts. 103, 159, 174 of the *Darlegung*.

‡ *M.N.R.A.S.* vol. xxxviii. pp. 43–49.

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Cowell.* The agreement with Delaunay's result is quite satisfactory. From the value of the latter there results

Daily Motion, −189″·7005.

12. *Terms of the Second Order.*—The results are exhibited in the same way as in § 8. The portion depending on γ^2 is extracted from Mr. Cowell's paper just referred to ; those depending on e^2, e'^2, are the results of my calculations.

Daily Motions.

Factors.	e^2	γ^2	e'^2
	″	″	″
m^2	−1·1999	+0·8021	−0·0840
m^3	−0·3534	− ·0675	+ ·0040
m^4	− ·0958	− ·0197	+ ·0053
m^5	− ·0270	− ·0016	+ ·0022
Sum	−1·6761	+0·7133	−0·0725
Est. Rem.	− ·010	+ ·000	+ ·0009
Delaunay	−1·686	+0·713	−0·072
True value	−1·6865	+0·7134	−0·0697

The agreement in all cases is sufficiently close. †

13. *Terms of the Fourth Order.*—As in the case of the perigee, we obtain these from Delaunay's expressions.

Daily Motions.

Factors.	e^4	γ^4	e'^4	a^2/a'^2
	″	″	″	″
m^2	+0·0008	[none]	−·0000	−·0024
m^3	− ·0005	+·0001	...	−·0005
Est. Rem.	− ·0003	...	...	−·0001
Delaunay	+ ·0000	+·0001	−·0000	−·0030

Daily Motions.

Factors.	$\gamma^2 e^2$	$\gamma^2 e'^2$	$e^2 e'^2$
	″	″	″
m^2	−·0103	+·0003	−·0005
m^3	+·0043	−·0001	−·0004
Est. Rem.	+·0030	−·0002	−·0004
Delaunay	−·0030	+·0000	−·0013

The sum of these gives

Daily Motion −0″·007,

with a possible error due to wrong estimation of ±″·003. Nearly the whole of this latter quantity arises in the estimation of the terms depending on $\gamma^2 e^2$.

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* *Amer. Jour. Math.* vol. xviii. pp. 99–127.

† See the Postscript to this paper.

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Annual Motion of the Node.

14. Proceeding as before, we find the following :—

Terms of the form $nf(m)$	−69 288″·1	(Adams & Cowell)	
,, ,, ,, $ne^2f(m)$	− 616·0	(Brown)	
,, ,, ,, $n\gamma^2f(m)$	+ 260·6	(Cowell)	
,, ,, ,, $ne'^2f(m)$	− 25·5	(Brown)	
,, ,, higher orders	− 2·6 ± 1″·1	(Delaunay)	
From solar action	−69 671·6 ± 1·1		
From the figure of the Earth	− 6·4	(Hill)	
From planetary action	− 1·5	(Hansen)	
From theory [†]	−69 679·5 ± 1·1	. . . .	(i)
From observation	−69 679·5	(Newcomb)[*] . . .	(ii)
From Hansen's theory	−69 676·8	. . . .	(iii)

Hansen obtains for the inequality arising from the figure of the Earth the value —5″·9. Accepting Dr. Hill's value, we find for Hansen's theoretical value

−69 677·3 . . . . (iv)

The close agreement between the results (i) which I have deduced from theory and (ii) deduced from observation must be considered in connection with the possible error ± 1″·1. The corrected result of Hansen (iv) still differs from that given by observation by 2″·2. It appears, then, that the new theoretical value is rather more accurate than that of Hansen. It is probable that when the remainder of the solar inequalities in the motion of the node have been determined, an almost perfect agreement will be found between theory and observation.

15. While this determination is in progress, it would seem from the above discussion that the most accurate theoretical values as yet obtained are the result (iv) in the case of the perigee and the result (i) in the case of the node; that is to say,

Annual Mean Motion of the Perigee	+146 434″·9,
Annual Mean Motion of the Node	− 69 679″·5,

the amount by which these values may differ from the correct

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[*] This value was kindly communicated to me by Professor Newcomb. It differs from Hansen's value by 0″·1.

[†] This result was announced at the December meeting of the American Mathematical Society.

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theoretical values being not more than $\frac{1}{70,000}$ part of the whole in each case.

Haverford College, Pa.:
1897 *January* 5.

Postscript.—In the course of obtaining a verification of Delaunay's expressions for the secular accelerations, an error in the *Théorie* was discovered in the following way. The value of the part of the motion of the node which depends on e'^2, as given by Delaunay, is

$$-ne'^2\left(\frac{9}{8}m^2-\frac{23}{32}m^3-\frac{3261}{256}m^4-\frac{73423}{1024}m^5\right).$$

This value failed to furnish correctly the terms in the secular acceleration of the node which have the factors m^3, m^5, and, in that of the mean motion, the factors $m^3\gamma^2$, $m^5\gamma^2$. At first the natural assumption was that these four terms were wrong; but, on reconsidering Delaunay's method of obtaining them, this did not seem probable. I found that *all* the terms could be made to agree if we had

$$-ne'^2\left(\frac{9}{8}m^2-\frac{33}{32}m^3-\frac{3261}{256}m^4-\frac{73413}{1024}m^5\right)$$

instead of the above expression.

On referring to de Pontécoulant, *Système du Monde*, vol. iv. p. 567, I found that he gave 33 instead of 23 as the numerator of the second term. Moreover, the small difference between my result and that of Delaunay, as shown in the last column of the table in § 12 above, is exactly accounted for when the change is made.

It then seemed worth while to see if an error in Delaunay's *Théorie* could be detected without much trouble, and it was almost immediately found to be a simple error of transcription. The term

$$m'\frac{a^2}{a'^3}\left\{\cdots\left(\cdots+\frac{33}{16}\gamma^2e'^2\cdots\right)\frac{n'}{n}\cdots\right\},$$

in the value of the constant part of R on p. 2 of vol. ii. is incorrectly transcribed

$$m'\frac{a^2}{a'^3}\left\{\cdots\left(\cdots+\frac{23}{16}\gamma^2e'^2\cdots\right)\frac{n'}{n}\cdots\right\},$$

in the more extended value on p. 234, and the error is retained in the values of $\frac{dl}{dt}$, $\frac{dg}{dt}$, $\frac{dh}{dt}$, which immediately follow. This accounts for the error in the second term of the expression given above, and doubtless for that in the fourth term also.